Multi-channel Queues with Setup Time

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Abstract

Many practical queuing situations with congestion control mechanism due to high throughput demands in telecommunication systems, computer network and production systems can be formulated as finite queues with setup time and state dependent arrivals. This chapter deals with computational scheme to compute the exact stationary queue length distribution. In this chapter an efficient iterative algorithm is developed for computing the stationary queue length distribution in M/G/K/N queues with setup time and arbitrary state dependent arrival rates. The overall computation of the algorithm is O(N2) is complexicity. It can be of great use in application since it is easy to implement fast and quite accurate.

1. Introduction

The arrival occur according to Poisson process which depends on the number of customers in the system. We consider a M/G/K/N queue with state-dependent arrivals and set up time. Which was discussed earlier by Courtois and Georges (1971). The server has a set up time before serving the first customer who initializes a busy period which was best explained by Baker (1973) for the queue M/M/1 with exponential startup. Gordan and Newell (1967) also studied the queueing system with exponential servers. The service process is assumed to be independent of any process in system. The system can hold upto N customers including the one under service at any point of time. The service discipline is exhaustive and FCFS was studied by Shantikumar and Sumita (1985) of M/G/1/K queues with state dependent arrivals and FCFS/LCFS-p service disciplines.

This model has a wide range of application in telecommunication systems, production system and inventory control. ATM (Asynchronous Transfer Mode) technique is now broadly accepted for constructing high speed multimedia communication networks which was again analyzed by Skelly et al. (1993) and a Histogram based Model for Video Traffic Behaviour in an ATM multiplexer was developed. Reiser (1982) studied performance evaluation of data communication systems and due to the high throughput demands, these networks usually employ simplified and universal congestion control mechanism which are based on input rate enforcement in order to provide and maintain good quality of service (Q_0S) Schmidt and Compbell (1993) also studied Protocol Traffic Analysis with application for ATM switch Design which was brought forth by Keshav et al. (1995) through the study of an empirical evaluation of virtual circuit holding policies in Ip-over ATM Network.

It is believed that in a densely connected network the aggregated arrival process to the intermediate node can be approximated by a Poisson process suggested by Kee, and Towsley, (1986). A cell set up phase is generally needed before starting each busy period. This motivated up to study M(n)/G/K/N queue with set up time.

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Mn/G/K/N queues have been given relatively little attention. Kijima and Makimoto (1992) give numerical algorithms to compute the quasi-stationary distribution and other characteristics in Mn/G/1/N queues and GI/M(n)/1/N queues byusing Matrix-geometric method.

Chaudhary, Gupta and Agarwal (1991) also examined computational analysis of distribution of numbers in system for M/G/1/N+1 and G/M/1/N+1 queues using roots. Gong et al (1992) also provide a numerical algorithm based on Matrix-geometric method of M/G/1 queues with state dependent arrivals. Recently, Yang proposes a new approach for computing the stationary queue length distribution in M(n)/G/1/N queues and GI/Mn/1/N queues. In this paper we develop an algorithm for computing the stationary queue length distribution in Mn/G/K/N queues with setup time by using the method of supplementary and variables. Buzen; (1973) also suggested computational Algorithms for closed Queueing Network with exponential servers. The rest of the paper is organized as follows. The section 2 we derive the system equations by using the method of supplementary variables. An interactive algorithm with overall. Computation $O(N^2)$ in complexity is developed for computing the stationary queue length distribution of Mn/G/1/N with setup.

2. System Euqations

Consider a M(n)/G/K/N queue with setup time described in section 1. Let λ_n be the arrival rate where there are n customers in the system. Since buffer size is N for $n \geq N$, $\lambda_n = 0$. It is assumed that $\lambda_n > 0$ for $0 \leq n \leq N-1$. The probability density function (pdf) of the service time and its corresponding Laplace transform (L.T.) are denoted by b (.) and B*(s), respectively. We denote the mean of the service time by Kμ.The pdf of the service time is deonted by a (.) with L.T. A*(s) and mean v. Let Q(t) be the number of customers in the system at time t. We define the

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supplementary variable U(t) as the remaining service time or the remaining setup time at time t. Let

$$R(t)=\begin{cases}1 & \text{server is busy or stays idle at time t;}\\ 0 & \text{server is setting up at time t.}\end{cases}$$

Clearly, the process {[Q(t), R(t), U(t)]; t ≥ 0} is a Markov chain. Define the steady state joint density functions of {[Q(t), R(t), U(t)]; t ≥ 0} as

$$\begin{cases}f_n(u)\Delta u=\lim_{r\to\infty} \quad \Pr ob\{Q(t)=n, R(t)=1, u<U(t)<u+\Delta u\};\\ g_n(u)\Delta u=\lim_{t\to\infty} \quad \Pr ob\{Q(t)=n, R(t)=0, u<U(t)<u+\Delta u\}.\end{cases}$$

Let Q be the number of customers in the system in steady state. Then, the stationary queue length distribution is given by P(Q=n), o ≤ n < ∞. By infinitestimal argument (idea given by Taylor and Karlin, (1994)), we have following steady state equations:

$$\lambda_0 P\,(Q=0) = f_1(0),$$

$$-\frac{df_1(u)}{du}=-\lambda_1 f_1(u)+f_2(0)b(u)+g_1(0)b(u),$$

$$-\frac{df_n(u)}{du}=+\lambda_{n-1}f_{n-1}(u)-\lambda_n f_n(u)+f_{n+1}(0)b(u)$$

$$+g_n(0)b(u), 2\le n\le N-1,$$

$$-\frac{df_N(u)}{du}=-\lambda_{N-1}(u)+g_N(0)b(u)$$

$$-\frac{dg_1(u)}{du}=\lambda_0 P(Q=0)a(u)-\lambda_1 g_1(u)$$

$$-\frac{dg_n(u)}{du}=\lambda_{n-1}g_{n-1}(u)-\lambda_n g_n(u), 2\le n\le N-1$$

$$-\frac{dg_N(u)}{du}=\lambda_{N-1}g_{N-1}(u)$$

Denote

$$P_n^*(s)=\int_0^\infty e^{-su}f_n(u)du, \qquad n = 1, 2... N,$$

$$q_n^*(s)=\int_0^\infty e^{-su}g_n(u)du, \qquad n = 1, 2... N$$

And $p_0^*(0)=P(Q=0)$. Since $p_0^*(0)$ and $p_n^*(0)+q_n^*(0)=P(Q=n)$ for 1 ≤ n ≤ N are the stationary queue length distribution, our objective is to determine $\{p_0^*(0), p_n^*(0), q_n^*(0), 1\le n\le N\}$.

Taking the Laplace transform of (1), we have

$$\lambda_0 p_0^*(0)=f_1(0),$$

$$(\lambda_1-s)p_1^*(s)=B^*(s)f_2(0)+B^*(s)g_1(0)-f_1(0),$$

$$(\lambda_n-s)p_n^*(s)=\lambda_{n-1}p_{n-1}^*(s)+B^*(s)f_{n+1}(0)+B^*(s)g_n(0)$$

$$-f_n(0),\ 2\le n\le N-1,$$

$$-sp_N^*(s)=\lambda_{N-1}p_{N-1}^*(s)+B^*(s)g_N(0)-f_N(0),$$

.... (2)

$$(\lambda_1-s)q_1^*(s)=\lambda_0 A^*(s)p_0^*(0)-g_1(0,)$$

$$(\lambda_n-s)q_n^*(s)-\lambda_{n-1}q_{n-1}^*(s)-g_n(0,)2\le n\le N-1,$$

$$-sq_N^*(s)=\lambda_{N-1}q_{N-1}^*(s)-g_N(0)$$

By substituting s=0 into equation (2), we can have following lemma which gives expression of $f_n(0)$ and $g_n(0)$ in terms of $p_n^*(0)$'s and $q_n^*(0)$'s after some algebraic manipulations.

Lemna 1

$$\begin{cases}f_1(0)=\lambda_0 p_0^*(0),\\ f_n(0)=\lambda_{n-1}\left[p_{n-1}^*(0)+q_{n-1}^*(0)\right],\ 2\le n\le N,\\ g_1(0)=\lambda_0 p_0^*(0)-\lambda_1 q_1^*(0),\\ g_n(0)=\lambda_{n-1}q_{n-1}^*(0)-\lambda_n q_n^*(0),\ 2\le n\le N-1,\\ g_N(0)=\lambda_{N-1}q_{N-1}^*(0.)\end{cases}$$

..... (3)

We can eliminate $f_n(0)$'s and $g_n(0)$'s in (2) by using Lemma 1.

$$\begin{cases}(\lambda_1-s)p_1^*(s)=\lambda_1 B^*(s)p_1^*(0)+\lambda_0(B^*(s)-1)p_0^*(0),\\ (\lambda_n-s)p_n^*(s)=\lambda_{n-1}\left[p_{n-1}^*(s)-P_{n-1}^*(0)\right]\\ +\lambda_n B^*(s)p_n^*(0)+\lambda_{n-1}(B^*(s)-1)q_{n-1}^*(0),\\ -Sp_N^*(s)=\lambda_{N-1}\left[p_{N-1}^*(s)-p_{N-1}^*(0)\right]+\lambda_{N-1}(B^*(s)-1)q_{N-1}^*(0),\\ \lambda_1-s(q_1^*(s)=\lambda_0[A^*(s)-1]p_0^*(0)+\lambda_1 q_1^*(0),\\ (\lambda_1-s)q_n^*(s)=\lambda_{n-1}\left[q_{n-1}^*(s)-q_{n-1}^*(0)\right]+\lambda_n q_n^*(0),\\ -Sq_N^*(s)=\lambda_{N-1}\left[q_{N-1}^*(s)-q_{N-1}^*(0)\right]\end{cases} \quad ...(4)$$

For 2 ≤ n ≤ N – 1. Setting s = λ_1 into the ith and the (N+1)th equations in (4) for i = 1, 2, …., N – 1 gives

$$p_1^*(0)=\frac{\lambda_0[1-B^*(\lambda_1)]p_0^*(0)}{\lambda_1 B^*(\lambda_1)}$$

$$q_1^*(0)=\frac{\lambda_0[1-A^*(\lambda_1)]p_0^*(0)}{\lambda_1}$$

$$p_n^*(0)=\frac{\lambda_{n-1}\left[p_{n-1}^*(0)-p_{n-1}^*(\lambda_n)\right]+\lambda_{n-1}(1-B*(\lambda_n)]q_{n-1}^*(0)}{\lambda_n B*(\lambda_n)}, \quad \text{.... (5)}$$

$$q_n^*(0)=\frac{\lambda_{n-1}\left[q_{n-1}^*(0)-q_{n-1}^*(\lambda_n)\right]}{\lambda_n}$$

for $2 \le n \le N-1$. We can also re-arrange some equations in (4) as

$$p_1^*(s)=\frac{\lambda_0[B*(s)-1]p_0^*(0)+\lambda_1 B*(s)p_1^*(0)}{\lambda_1 - s}$$

$$q_1^*(s)=\frac{\lambda_0[A*(s)-1]p_0^*(0)+\lambda_1 q_1^*(0)}{\lambda_1 - s}$$

$$p_n^*(s)=\frac{\lambda_{n-1}\left[p_{n-1}^*(s)-p_{n-1}^*(0)\right]+\lambda_n B*(s)p_n^*(0)+\lambda_{n-1}[B*(s)-1]q_{n-1}^*(0)}{\lambda_n - s}$$

$$q_n^*(s)=\frac{\lambda_{n-1}\left[q_{n-1}^*(s)-q_{n-1}^*(0)\right]+\lambda_n q_n^*(0)}{\lambda_n - s}\text{f}$$

for $2 \le n \le N-1$. Note that $p_n^*(s)$ and $q_n^*(s)$ for $n \ge 1$ in (6) are well-defined at $s = \lambda_n$ because both numerator and denominator of $p_n^*(s)$ and $q_n^*(s)$ for $n \ge 1$ have zero at $s = \lambda_n$. We denote (5) and (6) as the system equations. An iterative algorithm will be developed for computing the stationary queue length distributions $\left\{ p_n^*(0), p_n^*(0)+q_n^*(0), 1 \le n \le N \right\}$ based on these equations.

3. The Algorithm

In this section, we develop an efficient scheme for computing the stationary queue length distribution with overall computation $0(N^2)$ in complexity.

From the system equations (5) and (6), there exist $x_n(s)$ and $y_n(s)$ such that

$$p_n^*(s)=x_n(s)p_n^*(0),$$
$$q_n^*(s)=y_n(s)p_n^*(0), \quad \text{..... (7)}$$

for n = 1, 2, …., N. By the normalization condition

$$p_0^*(0)+\sum_{n=1}^{N}\left[p_n^*(0)+q_n^*(0)\right]=1$$ and (7),

Thus, from (7) and (8)

$$p_n^*(0)=\frac{x_n(0)}{1+\sum_{n=1}^{N}[x_n(0)+y_n(0)]}$$

$$q_n^*(0)=\frac{y_n(0)}{1+\sum_{n=1}^{N}[x_n(0)+y_n(0)]}$$

for $1 \le n \le N$. The next lemma provides a formula of $x_N(0) + y_N(0)$ in terms of $[x_N(0), y_N(0), 1 \le n \le N-1\}$.

Lemma 2

$$x_N(0)+y_N(0)=(k\mu+\nu)\lambda_0+\sum_{n=1}^{N-1}(\lambda_n k\mu-1)[x_n(0)+y_n(0)] \quad \text{..... (10)}$$

Proof

Adding equations in (4), we have

$$-s\sum_{n=1}^{N}\left[p_n^*(0)+q_n^*(0)\right]=$$
$$\lambda_0[B*(s)-1+A*(s)-1]p_0^*(0)+\sum_{n=1}^{N-1}\lambda_n[B*(s)-1]\left[p_n^*(0)+q_n^*(0)\right] \quad \text{....(6)}$$

That is,

$$\sum_{n=1}^{N}\left[p_0^*(0)+q_n^*(0)\right]=$$
$$\lambda_0\left[\frac{B*(s)-1}{-s}+\frac{A*(s)-1}{-s}\right]p_0^*(0)+\sum_{n=1}^{N-1}\lambda_n\left[\frac{B*(s)-1}{-s}\right]\left[p_n^*(0)+q_n^*(0)\right]$$

Observe that

$$-\left.\frac{dB*(s)}{ds}\right|_{s=0}=k\mu \quad \text{and} \quad -\left.\frac{dA*(s)}{ds}\right|_{s=0}=\nu$$

Letting $s \to 0$ in above equation gives

$$p_N^*(0)+q_N^*(0)=(k\mu+\nu)\lambda_0 p_0^*(0)+\sum_{n=1}^{N-1}(\lambda_n k\mu-1)\left[p_N^*(0)+q_N^*(0)\right]$$

The desired result follows immediately by using (7).
In addition, the system blocking probability is given by

$$P(Q=N)=p_N^*(0)+q_N^*(0)=\frac{x_N(0)+y_N(0)}{1+\sum_{n=1}^{N}[x_n(0)+y_n(0)]}$$

Therefore, from (8), (9) and (10), we only need to evaluate $\{x_n(0), y_n(0), 1 \le n \le N-1\}$ to determine the stationary queue length distribution

$$p_0^*(0)=\frac{1}{1+\sum_{n=1}^{N}[x_n(0)+y_n(0)]} \quad \text{.....(8)}$$

$\left\{ p_0^*(0), p_n^*(0)+q_n^*(0), 1 \le n \le N \right\}$.

We denote $x_0(s) = B*(s)$ and $y_0(s) = A*(s)$ for convenience. From (5), (6) and (7), we have

$$x_1(0)=\frac{\lambda_0[1-x_0(\lambda_1)]}{\lambda_1 x_0(\lambda_1)}$$

$$y_1(0)=\frac{\lambda_0[1-y_0(\lambda_1)]}{\lambda_1}$$

$$x_n(0)=\frac{\lambda_{n-1}[x_{n-1}(0)-x_{n-1}(\lambda_n)]+\lambda_{n-1}[1-x_0(\lambda_n)]y_{n-1}(0)}{\lambda_n x_0(\lambda_n)}$$

$$y_n(0)=\frac{\lambda_{n-1}[y_{n-1}(0)-y_{n-1}(\lambda_n)]}{\lambda_n}$$

$$x_1(s)=\frac{\lambda_0[x_0(s)-1]+\lambda_1 x_0(s)x_1(0)}{\lambda_1-s} \quad \text{..... (11)}$$

$$y_1(s)=\frac{\lambda_0[y_0(s)-1]+\lambda_1 y_1(0)}{\lambda_1-s}$$

$$x_n(s)=\frac{\lambda_{n-1}[x_{n-1}(s)-x_{n-1}(0)]+\lambda_n x_0(s)x_n(0)+\lambda_{n-1}[x_0(s)-1]y_{n-1}(0)}{\lambda_n-s}$$

$$y_n(s)=\frac{\lambda_{n-1}[y_{n-1}(s)-y_{n-1}(0)]+\lambda_n y_0(0)}{\lambda_n-s}$$

for $2 \le n \le N-1$. Observe that in order to obtain $\{x_n(0), y_n(0), 1 \le n \le N-1\}$, we still need to evaluate $\{x_{n-1}(\lambda_n), y_{n-1}(\lambda_n), 1 \le n \le N-1\}$. Let $w^{(i)}(s) = [d^i w(s)]/[ds^i]$. By some algebraic manipulations, one can have,

$$x_1^{(i)}(s)=\frac{[\lambda_0+\lambda_1 x_1(0)]x_0^{(i)}(s)+ix_1^{(i-1)}(s)}{\lambda_1-s}$$

$$y_1^{(i)}(s)=\frac{\lambda_0 y_0^{(i)}(s)+iy_1^{(i-1)}(s)}{\lambda_1-s},$$

$$x_n^{(i)}(s)=\frac{\lambda_{n-1}x_{n-1}^{(i)}(s)+[\lambda_n x_n(0)+\lambda_{n-1}y_{n-1}(0)]x_0^{(i)}(s)+ix_n^{(i-1)}(s)}{\lambda_n-s} \quad \text{..... (12)}$$

$$y_n^{(i)}(s)=\frac{\lambda_{n-1}y_{n-1}^{(i)}(s)+iy_n^{(i-1)}(s)}{\lambda_n-s}$$

for $2 \le n \le N-1$, $i \ge 1$.

Therefore, if $\lambda_k \ne \lambda_n$,

$$x_1(\lambda_k)=\frac{\lambda_0[x_0(\lambda_k)-1]+\lambda_1 x_0(\lambda_k)x_i(0)}{\lambda_1-\lambda_k}$$

$$y_1(\lambda_k)=\frac{\lambda_0[y_0(\lambda_k)-1]+\lambda_1 y_1(0)}{\lambda_1-\lambda_k}$$

$$x_n(\lambda_k)=\frac{\lambda_{n-1}[x_{n-1}(\lambda_k)-x_{n-1}(0)]+\lambda_n x_0(\lambda_k)x_n(0)+\lambda_{n-1}[x_0(\lambda_k)-1]y_{n-1}(0)}{\lambda_n-\lambda_k}$$

$$y_n(\lambda_k)=\frac{\lambda_{n-1}[y_{n-1}(\lambda_k)-y_{n-1}(0)]+\lambda_n y_n(0)}{\lambda_n-\lambda_k}$$

$$x_1^{(i)}(\lambda_k)=\frac{[\lambda_0+\lambda_1 x_1(0)]x_0^{(i)}(\lambda_k)+ix_1^{(i-1)}(\lambda_k)}{\lambda_1-\lambda_k}$$

$$y_1^{(i)}(\lambda_k)=\frac{\lambda_0 y_0^{(i)}(\lambda_k)+iy_1^{(i-1)}(\lambda_k)}{\lambda_1-\lambda_k}$$

$$x_n^{(i)}(\lambda_k)=\frac{\lambda_{n-1}x_{n-1}^{(i)}(\lambda_k)+[\lambda_n x_n(0)+\lambda_{n-1}y_{n-1}(0)]x_0^{(i)}(\lambda_k)+ix_n^{(i-1)}(\lambda_k)}{\lambda_n-\lambda_k}$$

$$y_n^{(i)}(\lambda_k)=\frac{\lambda_{n-1}y_{n-1}^{(i)}(\lambda_k)+iy_n^{(i-1)}(\lambda_k)}{\lambda_n-\lambda_k},$$

for $2 \le n \le N-1$, $i \ge 1$.

Otherwise, if $\lambda_k = \lambda_n$,

$$x_1^{(i)}(\lambda_k)=-\frac{[\lambda_0+\lambda_1 x_1(0)]x_0^{(i+1)}(\lambda_k)}{i+1}$$

$$y_1^{(i)}(\lambda_k)=-\frac{\lambda_0 y_0^{(i+1)}(\lambda_k)}{i+1}$$

$$x_n^{(i)}(\lambda_k)=-\frac{\lambda_{n-1}x_{n-1}^{(i+1)}(\lambda_k)+[\lambda_n x_n(0)+\lambda_{n-1}y_{n-1}(0)]x_0^{(i+1)}(\lambda_k)}{i+1} \quad \text{.... (14)}$$

$$y_n^{(i)}(\lambda_k)=-\frac{\lambda_{n-1}y_{n-1}^{(i+1)}(\lambda_k)}{i+1}$$

for $2 \le n \le N-1$, $i \ge 0$.

However, it is not necessary to evaluate all $\{x_n^{(i)}(\lambda_k), y_n^{(i)}(\lambda_k)$, $0 \le n \le N-1$, $0 \le i \le N-1$, $1 \le k \le N-1\}$ to calculate $\{x_n(0), y_n(0), 1 \le n \le N-1$. An efficient scheme is developed in the following.

For simplicity, we may assume that the arrival rates can be divided into m groups based on m + 1 threshold values $N_0 = 0 < N_1 < N_2 < \ldots < N_m = N$ such that

$$\lambda_i = \begin{cases} \lambda_{N_0} & 0 \le i < N_1 \\ \lambda_{N_1} & N_1 \le i < N_2 \\ \dots & \dots \\ \lambda_{N_{m-1}} & N_{m-1} \le i < N_m = N \\ 0 & i \ge N, \end{cases}$$

for $i \le 0$, where $\lambda_{Ni_1} \ne \lambda_{Ni_2}$ if $i_1 \ne i_2$. This assumption is very practical, although it is not hard to modify the following algorithm to the general cases.

Let l(k) = max $\{1, N_{i_k}\}$ such that $N_{i_k} \le k \le N_{i_{k+1}}$ is the least positive number with $\lambda_{l(k)} = \lambda_k$. Denote $L_n(k)$ as the number of λ_i such that $\lambda_i = \lambda_k$ for $n + 1 < i \le k$, that is, $L_n(k) = k - \max[n+1, l(k)]$ for $0 \le n \le k - 1$ and $2 \le k \le N - 1$. From above definitions, we immediately have following lemma :

Lemma 3

For $0 \le n \le k - 1$, $2 \le k \le N - 1$,

(1) If $L_n(k) \ge 1$, $\lambda_k = \lambda_{k-1}$ and $L_n(k) = Ln(k + 1)+1$.

(2) if $\lambda_n \ne \lambda_k$, $L_n(k) = L_{n-1}(k)$.

(3) if $\lambda_n = \lambda_k$, $L_n(k) = L_{n-1}(k)-1$.

Lemma 4

If $I = L_{n-1}(k) + l(k)$ for $1 \le n \le k - 1$, $2 \le k \le N - 1$, then $x_0^{[L_{n-1}(k)]}(\lambda_k) = x_0^{[L_0(I)]}(\lambda_1)$

Proof :

$I = L_{n-1}(k) + l(k) = k\text{-}\max[n, l(k)] + l(k) \le k$.

On the other hand, $I = L_{n-1}(k) + l(k) \ge l(k)$. Thus, by assumption on the arrival rates, $\lambda_1 = \lambda_k$. Since we always have $l(I) \ge 1$,

$L_0(I) = I - \max[1, l(I)] = I - l(I) = I - l(k) = L_{n-1}(k)$.

We arrive at the desired result.

Using Lemma 3 and lemma 4, (13) and (14) can be written as following:

For $\lambda_n \ne \lambda_k$, $1 \le n \le k - 1$, $2 \le k \le N - 1$.

if $L_n(k) = 0$

$$x_1^{[L_1(k)]}(\lambda_k) = \frac{\lambda_0\left\{x_0^{[L_0(k)]}(\lambda_k)\right\} - 1) + \lambda_1 x_0^{[L_0(k)]}(\lambda_k) x_1(0)}{\lambda_1 - \lambda_k}$$

$$y_1^{[L_1(k)]}(\lambda_k) = \frac{\lambda_0\left\{y_0^{[L_0(k)]}(\lambda_k) - 1\right\} + \lambda_1 y_1(0)}{\lambda_1 - \lambda_k}$$

$$x_b^{[L_n(k)]}(\lambda_k) = \frac{\lambda_{n-1}\left\{x_{n-1}^{[L_{n-1}(k)]}(\lambda_k) - x_{n-1}(0)\right\} + \lambda_n x_0^{[L_0(I)]}(\lambda_1) x_n(0) + \lambda_{n-1}\left\{x_0^{[L_0(I)]}(\lambda_1) - 1\right\} y_{n-1}(0)}{\lambda_n - \lambda_k}$$

$$y_n^{[L_n(k)]}(\lambda_k) = -\frac{\lambda_{n-1}\left[y_{n-1}^{[L_{n-1}(k)]}(\lambda_k) - y_{n-1}(0)\right] + \lambda_n y_0(0)}{\lambda_n - \lambda_k}$$

if $L_n(k) \ge 1$,

$$x_1^{[L_1(k)]}(\lambda_k) = \frac{[\lambda_0 + \lambda_1 x_1(0)] x_0^{[L_0(k)]}(\lambda_k) + L_1(k) x_1^{[L_1(k-1)]}(\lambda_{k-1})}{\lambda_1 - \lambda_k}$$

$$x_n^{[L_n(k)]}(\lambda_k) = \frac{\lambda_{n-1} x_{n-1}^{[L_{n-1}(k)]}(\lambda_k) + [\lambda_n x_n(0) + \lambda_{n-1} y_{n-1}(0)] x_0^{[L_0(I)]}(\lambda_1) + L_n(k) x_n^{[L_n(k-1)]}(\lambda_{k-1})}{\lambda_n - \lambda_k}$$

$$y_n^{[L_n(k)]}(\lambda_k) = \frac{\lambda_{n-1} y_{n-1}^{[L_n(k)]}(\lambda_k) + L_n(k) y_n^{[L_n(k-1)]}(\lambda_{k-1})}{\lambda_1 - \lambda_k}$$

for $\lambda_n = \lambda_k$, $1 \le n \le k - 1$, $2 \le k \le N - 1$.

$$x_1^{[L_1(k)]}(\lambda_k) = -\frac{[\lambda_0 + \lambda_1 x_1(0)] x_0^{[L_0(k)]}(\lambda_k)}{L_1(k) + 1}$$

$$x_n^{[L_n(k)]}(\lambda_k) = -\frac{\lambda_{n-1} x_{n-1}^{[L_{n-1}(k)]}(\lambda_k) + [\lambda_n x_n(0) + \lambda_{n-1} y_{n-1}(0)] x_0^{[L_0(I)]}(\lambda_1)}{L_n(k) + 1}$$

$$y_n^{[L_n(k)]}(\lambda_k) = -\frac{\lambda_{n-1} y_{n-1}^{[L_{n-1}(k)]}(\lambda_k)}{L_n(k) + 1}$$

Observe that we only need to evaluate to $\left\{x_n^{[L_n(k)]}\lambda_k, y_n^{[L_n(k)]}\lambda_k\right.$, $0 \le n \le k - 1$, $2 \le k \le N - 1\}$ to obtain the stationary queue length distribution. In the following algorithm, e use $x_n(k)$, $y_n(k)$ to store $x_n^{[L_n(k)]}, y_n^{[L_n(k)]}(\lambda_k)$ respectively.

Algorithm

Step 1:

Given $\{\lambda_n, 0 \le n \le N - 1\}$ and kμ, ν.

For $1 \le k \le N - 1$, find i_k $N_{i_{k_1}} \le k < N_{i_{k+1}}$ such that and $l(k) \leftarrow \max\{1, N_{i_k}\}$

$$x_1(0) \leftarrow \frac{\lambda_0[1 - B^*(\lambda_1)]}{\lambda_1 B^*(\lambda_1)}$$

$$y_1(0) \leftarrow \frac{\lambda_0[1 - A^*(\lambda_1)]}{\lambda_1}$$

Step 2: .....(15)

For k = 2, 3… N – 1, do

(a) For n = 0, 1… k – 1, do

$L_n(k) \leftarrow k - \max[n+1, l(k)]$.
If $n \geq 1$, $l \leftarrow L_{n-1}(k) + l(k)$.
If n = 0 then

$$x_n(k) \leftarrow B*^{[L_n(k)]}(\lambda_k)$$

$$y_n(k) \leftarrow A*^{[L_n(k)]}(\lambda_k)$$

Else if $\lambda_n = \lambda_k$ then

If n = 1, $x_1(k) \leftarrow -\dfrac{[\lambda_0 + \lambda_1 x_1(0) x_0(k)]}{L_1(k)+1}$

Else

$$x_n(k) \leftarrow \frac{\lambda_{n-1}x_{n-1}(k) \; [\lambda_n x_n(0) + \lambda_{n-1}y_{n-1}(0)]x_0(I)}{L_1(k)+1}$$

Else if $L_n(k) \geq 1$ then

If n = 1,

$$x_1(k) \leftarrow \frac{[\lambda_0 + \lambda_1 x_1(0)]x_0(k) + L_1(k)x_1(k-1)}{\lambda_1 - \lambda_k}$$

Else

$$x_n(k) \leftarrow \frac{\lambda_{n-1}x_{n-1}(k)[\lambda_n x_n(0) + \lambda_{n-1}y_{n-1}(0)] \; x_0(I) + L_n(k)x_n(k-1)}{\lambda_n - \lambda_k}$$

$$y_n(k) \leftarrow \frac{\lambda_{n-1}y_{n-1}(k) + L_n(k)y_n(k-1)}{\lambda_n - \lambda_k}$$

Else if

n=1, $x_1(k) \leftarrow \dfrac{\lambda_0[x_0(k)-1] + \lambda_1 x_0(k)x_1(0)}{\lambda_1 - \lambda_k}$

$$y_1(k) \leftarrow \frac{\lambda_0[y_0(k)-1] + \lambda_1 y_1(0)}{\lambda_1 - \lambda_k}$$

Else

$$x_n(k) \leftarrow \frac{\lambda_{n-1}[x_{n-1}(k) - x_{n-1}(0)] + \lambda_n x_0(l)xn(0) \; + \lambda_{n-1}(x_0(I)-1)y_{x-1}(0)}{\lambda_n - \lambda_k}$$

$$y_n(k) \leftarrow \frac{\lambda_{n-1}[y_{n-1}(k) - y_{n-1}(0)] + \lambda_n y_n(0)}{\lambda_n - \lambda_k}$$

End (n).
(b)

$$x_k(0) \leftarrow \frac{\lambda_{k-1}[x_{k-1}(0) - x_{k-1}(k)] + \lambda_{k+1}(1 - B*(\lambda_k))y_{k-1}(0)}{\lambda_k B*(\lambda_k)}$$

$$y_k(0) \leftarrow \frac{\lambda_{k-1}[y_{k-1}(0) - y_{k-1}(k)]}{\lambda_k}$$

End (k).

Step 3: $x_N(0) + y_N(0) \leftarrow \lambda_0(k\mu + v) + \sum_{k=1}^{N-1}(\lambda_k k\mu - 1)[x_k(0) + y_k(0)]$

Step 4: $P(Q=0) \leftarrow \dfrac{1}{1 + \sum_{k=1}^{N}[x_k(0) + y_k(0)]}$

Step 5: $P(Q = k) \rightarrow [x_n(0)+y_k(0)] \, P(Q = 0)$ for k = 1, 2, …., N.

The above algorithm can be simplified further for the following two species cases of M(n)/G/k/N with setup time and state dependent arrivals :

1. For M/G/k/N queues with setup time with arrival rates $\lambda_n=\lambda$, we have $l(k) = 1$ and $L_n(k) - n + 1$.
2. For M(n)/G/k/N queues with setup time and distinct arrival rates, i.e. $\lambda_i \neq \lambda_j$ for $i \neq j$, we have $l(k) = k$ and $L_n(k) = 0$. Two well-known examples for this case are the 'discouragement' mechanism where $\lambda_n=\lambda/(n+1)$ and the machine interference problem where $\lambda_n=(N - n)\lambda$.

Further we can use the above algorithm to obtain the stationary queue length distribution for M(n)/G/k/N queues by letting ν = o and A*(s) = 1.

4. Numerical Result

In this section we use the algorithm to obtain the stationary queue length distribution in four M(n)/G/k/N queueing systems with or without setup times. The results for M(n)/E_5/K/N without setup time for N $\in$ {10, 20, 30). The arrival rate is λ_n = N – n if there are n customers in the system for $0 \leq n \leq N$. The other parameters for the algorithm are mean service time $\mu = \dfrac{1}{15}$, the mean setup time ν=0 and A*(s) = 1. It is compared with the result given by Kijima and Makimoto (1992).

Numerical result for M(n)/E_5/k/N queues with set up time and N $\in$ {10, 20, 3}. The arrival and the mean service time are the same while two types of set up time are chosen to test. The first one is exponential distribution with mean 1/30 and the other one is Erlang distribution with 2 phases and mean 1/30.

The results are compared by result given by Gong et al. (1992).

5. Conclusion

Two types of set up time are considered in this. The first queue has an exponential set up time, with mean 0.1 while the second queue has deter ministic setup time 0.1. In conclusion, the algorithm is powerful for general cases and it is easy to implement, fast and quite accurate.

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